Brachistochrone Problem

Find the shape of the curve down which a bead sliding from rest and accelerated by gravity will slip (without friction) from one point to another in the least time. The term derives from the Greek $\beta\rho\alpha\chi\iota\sigma\tau o\varsigma$ (*brachistos*) "the shortest" and $\chi\rho o\nu o\varsigma$ (*chronos*) "time, delay."

The brachistochrone problem was one of the earliest problems posed in the calculus of variations. Newton was challenged to solve the problem in 1696, and did so the very next day. In fact, the solution, which is a segment of a cycloid, was found by Leibniz, L'Hospital, Newton, and the two Bernoullis. Johann Bernoulli solved the problem using the analogous one of considering the path of light refracted by transparent layers of varying density. Actually, Johann Bernoulli had originally found an incorrect proof that the curve is a cycloid, and challenged his brother Jakob to find the required curve. When Jakob correctly did so, Johann tried to substitute the proof for his own.

In the solution, the bead may actually travel uphill along the cycloid for a distance, but the path is nonetheless faster than a straight line (or any other line).

The time to travel from a point P_1 to another point P_2 is given by the integral

$$t_{12} = \int_{P_1}^{P_2} \frac{ds}{v} ,$$

where s is the arc length and v is the speed. The speed at any point is given by a simple application of conservation of energy equating kinetic energy to gravitational potential energy,

$$\frac{1}{2}mv^2 = mgy$$

giving

$$v = \sqrt{2gy}$$

then $ds^2 = dx^2 + dy^2 \rightarrow ds = \sqrt{dx^2 + dy^2}$

now multiply the radical (above right) by $\frac{dx}{\sqrt{dx^2}}$

to get $$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + y'^2}\ dx$$

Back to our ‘old’ distance formula (d=rt) in fancy differential form:

$$t_{12} = \int_{P_1}^{P_2} \frac{ds}{v} = \int_{P_1}^{P_2} \frac{\sqrt{1+y'^2}\ dx}{\sqrt{2gy}}$$

What we’re trying to find is not the minimum time but rather the function (say F) which yields that minimum time… *A cool video shows a solution which takes this from here but (tricky) integrates with respect to ‘dy’.
URL: https://www.youtube.com/watch?v=geBj865enOg&t=9s “Ya gotta luv Youtube!”

The guy writes it all out so I’m going to stop here! The work above has similar steps but doesn’t use his trick(s). Have to admit, he loses me (could be his explanation?) after about the first 10 minutes of 18 total minutes.